A15271W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

Trinity Term 2016

MONDAY, 18 APRIL 2016, 14.30 to 16.30

You should submit answers to both of the two questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. **Magnetohydrodynamics.** In a version of magnetohydrodynamics often used to describe magnetised, weakly collisional plasmas, the pressure gradient in the momentum equation is replaced with the divergence of a pressure tensor:

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \equiv \rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla \cdot \mathsf{P} + \frac{\mathbf{j} \times \mathbf{B}}{c}, \tag{1}$$

where ρ is density, $\mathbf{u}$ velocity, $\mathbf{j}$ current density, and $\mathbf{B}$ magnetic field. The pressure tensor is $\mathsf{P} = p_\perp(\mathsf{I} - \mathbf{bb}) + p_\| \mathbf{bb}$, where $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ is the unit vector in the direction of the (local, instantaneous) magnetic field, I is the unit matrix and $p_\perp$ and $p_\|$ are scalar perpendicular and parallel pressures, respectively. These pressures satisfy certain evolution equations, the explicit form of which will not be required here. All other fields (ρ, $\mathbf{B}$, $\mathbf{j}$) satisfy the same equations as they do in standard MHD.

(a) [5 marks] Show that the rate of change of momentum in this approximation can be written in terms of modified total pressure and modified Maxwell stress (tension force) as follows

$$\rho \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\nabla \left(p_\perp + \frac{B^2}{8\pi} \right) + \nabla \cdot \left[\mathbf{bb} \left(p_\perp - p_\| + \frac{B^2}{4\pi} \right) \right]. \tag{2}$$

Comment on the physical effect that you expect positive or negative pressure anisotropy $p_\perp - p_\|$ to have on the motion of the fluid.

(b) [12 marks] Given a static, spatially homogeneous equilibrium state with constant $\rho = \rho_0$, $p_\perp = p_{\perp 0}$, $p_\| = p_{\|0}$, $\mathbf{u} = 0$, $\mathbf{B} = B_0 \hat{\mathbf{z}}$ ($\hat{\mathbf{z}}$ is the unit vector in the z direction), consider its infinitesimal perturbations $\delta\rho$, $\delta p_\perp$, $\delta p_\|$, $\mathbf{u}$ and $\delta\mathbf{B}$, assuming they are all of the form $\propto \exp(-i\omega t + i\mathbf{k} \cdot \mathbf{r})$, where $\mathbf{k} = (k_\perp, 0, k_\|)$. Show that these perturbations satisfy

$$-\omega \rho_0 \mathbf{u} = -k_\perp \hat{\mathbf{x}} \left(\delta p_\perp + \frac{B_0 \delta B}{4\pi} \right) - k_\| \hat{\mathbf{z}} \left[\delta p_\| + (p_{\perp 0} - p_{\|0}) \frac{\delta B}{B_0} \right] + \delta\mathbf{b}\, k_\| \left(p_{\perp 0} - p_{\|0} + \frac{B_0^2}{4\pi} \right), \tag{3}$$

where $\hat{\mathbf{x}}$ is the unit vector in the x direction and δB and $\delta\mathbf{b}$ are the perturbed field strength and direction, respectively. While in general, besides the induction equation, one also needs equations for $\delta p_\perp$ and $\delta p_\|$ to close the above equation, show that Alfvénic perturbations decouple and do not depend on $\delta p_\perp$ and $\delta p_\|$. Derive their dispersion relation and describe/sketch the fields and displacements associated with them.

(c) [3 marks] Under what condition do the Alfvénic perturbations become unstable? This is called the *firehose instability*. In the intergalactic medium, the typical pressure anisotropy is $|p_\perp - p_\||/p_\| \sim 10^{-2}$ and plasma beta $\beta = 8\pi p_\|/B^2 \sim 10^2$ (or larger). In (certain parts of) fusion devices, $|p_\perp - p_\||/p_\| \sim 10^{-1}$ and $\beta \sim 10^{-2}$ (all these numbers are order-of-magnitude). Which of these plasmas is likely to suffer from the firehose instability?

(d) [5 marks] Can you explain the physical mechanism of the firehose instability? What changes in the feedback to fluid displacements make Alfvénic perturbations in a pressure-anisotropic plasma described by the above equations unstable, while when $p_{\perp 0} = p_{\|0}$, these perturbations behave as propagating waves? Do firehose perturbations propagate as well as grow?

2. **Complex Fluids.**

(a) [5 marks] Show that a material vector field $\boldsymbol{\ell}$ in an incompressible fluid evolves according to

$$\frac{\partial \boldsymbol{\ell}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\ell} = \boldsymbol{\ell} \cdot \nabla \mathbf{u}.$$

Hence explain why the upper convected derivative

$$\frac{\partial \mathsf{A}}{\partial t} + \mathbf{u} \cdot \nabla \mathsf{A} - \mathsf{A} \cdot \nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathsf{T}} \cdot \mathsf{A},$$

with the convention that $[\mathsf{A} \cdot \nabla \mathbf{u}]_{ij} = A_{ik} \partial_k u_j$, is a suitable time derivative for a second-rank symmetric tensor A.

(b) The evolution of the stress T in a certain type of incompressible viscoelastic fluid with density ρ_0 is described by the equation

$$\mathsf{T} + \tau \left[\frac{\partial \mathsf{T}}{\partial t} + \mathbf{u} \cdot \nabla \mathsf{T} - \mathsf{T} \cdot \nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathsf{T}} \cdot \mathsf{T} \right] + \alpha \left(\mathsf{T} \cdot \mathsf{E} + \mathsf{E} \cdot \mathsf{T} \right) = 2\mu \mathsf{E}, \qquad (\dagger)$$

where α, τ and μ are constants, and $\mathsf{E} = \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathsf{T}} \right)$.

(i) [2 marks] Give a physical interpretation of the constants τ and μ.

(ii) [4 marks] Show that small-amplitude flows with $\mathbf{u} = u(y,t)\hat{\mathbf{x}}$ are governed by the linearised equations

$$\rho_0 \frac{\partial u}{\partial t} = \frac{\partial T}{\partial y}, \quad T + \tau \frac{\partial T}{\partial t} = \mu \frac{\partial u}{\partial y},$$

where T is a component of the tensor T that you should determine.

(iii) [5 marks] Hence show that linear waves proportional to $\exp(iky + \sigma t)$ exist with complex frequencies

$$\sigma = -\frac{1}{2\tau} \left[1 \pm (1 - 4\tau\nu k^2)^{1/2} \right],$$

where ν is a constant that you should determine. Interpret the limiting behaviour(s) as $\tau \to 0$.

(c) Now consider steady, but not necessarily small-amplitude, shear flows of the form $\mathbf{u} = \dot{\gamma}\, y\, \hat{\mathbf{x}}$ in the same fluid.

(i) [4 marks] Find the diagonal components of the stress tensor T, and show that the ratio of the normal stress differences $N_1 = T_{xx} - T_{yy}$ and $N_2 = T_{yy} - T_{zz}$ is

$$\frac{N_2}{N_1} = -\frac{\alpha}{2\tau}.$$

(ii) [5 marks] Find the shear stress T_{xy} and give a sketch of its behaviour as a function of $\dot{\gamma}$, identifying the limiting behaviours when $\dot{\gamma}$ is small and when $\dot{\gamma}$ is large. What kind of viscoelastic fluid is described by the equation ($\dagger$) above, and what condition should α satisfy?