A15758W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM MATTER

Trinity Term 2023

Thursday, 20th April, 9:30am - 11:30 am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider mass m spinless bosons in three dimensions in an applied potential $V(\mathbf{r})$. Suppose N identical such bosons are in the same potential and the bosons interact via a two-body repulsive potential $U(\mathbf{r}-\mathbf{r}') = U_0\,\delta(\mathbf{r}-\mathbf{r}')$, with δ being the three-dimensional Dirac delta function and $U_0 > 0$.

(a) Using canonical creation operators $\hat{\psi}^\dagger(\mathbf{r})$ and annihilation operators $\hat{\psi}(\mathbf{r})$ such that $[\hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}')] = \delta(\mathbf{r}-\mathbf{r}')$, write the Hamiltonian for this system in second quantized notation. A detailed derivation is not required. [3]

(b) Given an orthonormal complete basis of single-particle orbitals $\varphi_n(\mathbf{r})$, consider corresponding creation operators $\hat{a}_n^\dagger$ which create bosons in the orbitals φ_n where $[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}$. Express the field operator $\hat{\psi}^\dagger(\mathbf{r})$ in terms of the $\hat{a}_n^\dagger$. [3]

Now consider the case where the potential $V(\mathbf{r})$ is a symmetric double-well potential ($V(x,y,z) = V(-x,y,z)$ and there are minima of V at positions $x = \pm d, y = 0, z = 0$). For a *single boson* in this potential, the ground state orbital φ_0 with energy E_0 and the first excited state orbital φ_1 with energy E_1 can be accurately approximated as

$$\begin{aligned}\varphi_0(\mathbf{r}) &= [\phi_L(\mathbf{r}) + \phi_R(\mathbf{r})]\,/\sqrt{2}\\ \varphi_1(\mathbf{r}) &= [\phi_L(\mathbf{r}) - \phi_R(\mathbf{r})]\,/\sqrt{2}\end{aligned}$$

where $\phi_L(\mathbf{r})$ and $\phi_R(\mathbf{r})$ are two real normalized orthogonal orbitals localized near the minimum of the left and right wells respectively. The two orbitals $\phi_{L,R}$ have extremely small spatial overlap with each other which you may approximate as zero. Let $\hat{a}_L^\dagger$ and $\hat{a}_R^\dagger$ be creation operators for the left and right orbitals.

(c) Approximate $\psi^\dagger(\mathbf{r})$ only using $\hat{a}_L^\dagger$ and $\hat{a}_R^\dagger$, throwing away all other $\hat{a}_n^\dagger$ from the construction of part (b) above. In what limit will this approximation be valid? Show that the Hamiltonian for the many-boson system can be written in the Bose-Hubbard form

$$H = -t(\hat{a}_L^\dagger \hat{a}_R + \hat{a}_R^\dagger \hat{a}_L) + w(\hat{n}_L^2 + \hat{n}_R^2) - \mu(\hat{n}_L + \hat{n}_R)$$

where $\hat{n}_j = \hat{a}_j^\dagger \hat{a}_j$ with $j = L, R$. Give expressions for the values of the constants t, w and μ. Hint: you will need the fact that ϕ_L and ϕ_R have no spatial overlap. [9]

(d) Consider a normalized orbital of the form

$$\phi_{(\theta,\chi)}(\mathbf{r}) = \cos\theta\,\phi_L(\mathbf{r}) + \sin\theta\, e^{i\chi}\phi_R(\mathbf{r}).$$

Assuming there are a large number N of bosons in this double-well potential at low temperature, write a normalized many-body trial wavefunction which would correspond to condensing all of the bosons in the orbital $\phi_{(\theta,\chi)}$. Calculate the expectation value of the Hamiltonian in this trial state as a function of θ and χ. What values of θ and χ give the lowest energy? [5]

(e) Show that

$$\frac{\partial}{\partial\chi}\langle H\rangle = i\langle [H, \hat{n}_L]\rangle$$

where the expectations are taken in the trial wavefunction $\phi_{(\theta,\chi)}$. Give a physical interpretation of this equality. [5]

2. Consider a gas of N interacting spinless fermions of mass m in two dimensions at zero temperature.

(a) Assuming the system starts in the ground state, calculate the total energy change of the system if we boost the Fermi surface by momentum p_0 in the $\hat{x}$ direction. [3]

(b) Due to interactions, the energy $\epsilon_{\mathbf{p}}$ of a single quasiparticle with momentum $\mathbf{p}$ near the Fermi surface is renormalized to

$$\epsilon_{\mathbf{p}} - \mu \approx (p_F/m^*)(|\mathbf{p}| - p_F)$$

with m^* being the effective mass, p_F the Fermi momentum and μ the chemical potential. Calculate the density of states for these quasiparticles near the Fermi surface. [4]

(c) We can assume that the quasiparticle density $n(\mathbf{p})$ is a step function in every direction, but we allow for the position of the step to depend on the angle $\theta_{\mathbf{p}}$ of the momentum $\mathbf{p}$

$$n(\mathbf{p}) = \Theta(\,|\mathbf{p}| - p_F - \nu(\theta_{\mathbf{p}})\,)$$

where $\Theta(x) = 1$ for $x < 0$ and $\Theta(x) = 0$ otherwise. Here $\nu(\theta)$ is a function that tells us how the Fermi surface shape has been deformed, where $\nu(\theta) = 0$ in the ground state. What is the form of $\nu(\theta)$ if we boost the entire Fermi surface by a very small momentum p_0 in the $\hat{x}$ direction? [3]

Following the Landau Fermi-liquid approach we write a free-energy functional for the system as follows

$$\mathcal{F} = \mathcal{F}_0 + \sum_{\mathbf{p}} [\epsilon(\mathbf{p}) - \mu]\, \delta n(\mathbf{p}) + \frac{1}{2} \sum_{\mathbf{p},\mathbf{p}'} f(\theta_{\mathbf{p}} - \theta_{\mathbf{p}'})\, \delta n(\mathbf{p})\, \delta n(\mathbf{p}')$$

where $\delta n(\mathbf{p}) = n(\mathbf{p}) - n_0(\mathbf{p})$ is the change in occupancy of the momentum $\mathbf{p}$ quasiparticle state compared with its occupancy in the ground state and $f(\theta) = f(-\theta)$ is a phenomenological interaction function.

(d) Explain briefly why the form of the Landau Fermi-Liquid functional is sufficient to describe all low energy processes. (Your answer should be a short paragraph or less.) [5]

(e) Explain how m and m^* are related by the free-energy functional. Substitute the form of $\nu(\theta)$ from part (c) into the Landau free-energy functional to derive the energy cost of a boost. Compare this expression to the result of sections (a) and (b) to derive an explicit relationship between f, m, m^*, and the quasiparticle density of states. [10]