A15090W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GROUPS AND REPRESENTATION

Hilary Term 2023

FRIDAY, 13TH JANUARY 2023, 09:30am to 12:30 pm

You should submit answers to ***three*** *out of the four questions.*

You have **3 hours** *writing time to complete the paper.*
The use of a calculator and/or computer algebra packages is **not** *allowed.*

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [4 marks] Define the terms 'sub-group', 'normal sub-group', 'representation' and 'irreducible representation' (in the following also called 'irrep').

   (b) [5 marks] For a group G and a sub-group $H \subset G$, a representation $R : G \to \mathrm{GL}(V)$ induces a representation $\tilde{R} : H \to \mathrm{GL}(V)$, by restriction from G to H. Explain the term 'branching' and why $\tilde{R}$ can be reducible even though R is not. Give an example (by choosing suitable groups G and H and a representation R) for the phenomenon of an irreducible representation branching into a reducible representation.

   (c) [5 marks] Explain how the cyclic groups $\mathbb{Z}_n$ can be viewed as sub-groups of $U(1)$. Find the branching of the irreducible representations of $U(1)$ under $\mathbb{Z}_n$.

   (d) [6 marks] Consider a field theory with a $U(1)$ symmetry and N complex scalar fields ϕ_i with $U(1)$ charges $\hat{Q}_i$, where $i = 1, \ldots, N$. For a sub-group $\mathbb{Z}_n \subset U(1)$, what are the $\mathbb{Z}_n$ charges $\hat{q}_i$ of the fields ϕ_i? Now suppose that the $U(1)$ symmetry is spontaneously broken by vacuum expectation values $\langle \phi_i \rangle \neq 0$. Determine under which conditions an unbroken $\mathbb{Z}_n$ sub-group (where $n > 1$) can be retained and express the largest possible n for such unbroken $\mathbb{Z}_n$ symmetries in terms of the charges $\hat{Q}_i$.

   (e) [5 marks] Consider a four-dimensional (relativistic) field theory with $U(1)$ (gauge) symmetry and M Weyl fermions ψ_a with $U(1)$ charges Q_a, where $a = 1, \ldots, M$. What are the charges q_a of these fermions under a sub-group $\mathbb{Z}_n \subset U(1)$? Write down the condition for this $U(1)$ symmetry to be anomaly-free. What does this condition imply for the $\mathbb{Z}_n$ charges q_a?

2. Consider the permutation group $S_4 = \{\sigma : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\} \,|\, \sigma \text{ bijective}\}$.

   (a) [5 marks] List the conjugacy classes of S_4 (in terms of partitions of 4), determine the number of permutations in each class and explicitly provide one permutation per class. How many irreducible representations (over complex vector spaces) does S_4 have?

   (b) [4 marks] Write down explicitly two one-dimensional irreps R_0 and R_1 of S_4 and their characters χ_0 and χ_1. What are the dimensions of the other irreducible representations of S_4?

   (c) [8 marks] Consider the maps $R_\pm : S_4 \to \mathrm{GL}(\mathbb{C}^4)$ defined by $R_+(\sigma)(e_i) = e_{\sigma(i)}$ and $R_-(\sigma)(e_i) = \mathrm{sgn}(\sigma) e_{\sigma(i)}$. Here, $\sigma \in S_4$ is a permutation, e_i, where $i = 1, 2, 3, 4$, are the standard unit vectors of $\mathbb{C}^4$ and sgn is the sign function for permutations. Show that $R_\pm$ are representations and compute their characters $\chi_\pm$. Show that each of $R_\pm$ contains one one-dimensional and one three-dimensional irrep. Find the character χ_3 of the three-dimensional irrep R_3 in R_+ and the character χ_4 of the three-dimensional irrep R_4 in R_-.

   (d) [4 marks] Based on the information collected so far, write down the character table of S_4.

   (e) [4 marks] Consider a scalar field $\phi = (\phi_1, \phi_2, \phi_3)^T$ which transforms under the three-dimensional S_4 irrep R_3 from part (c). Show that it is possible to write down S_4 invariant quadratic and cubic terms in ϕ.

3. (a) [4 marks] State Schur's Lemma.

   (b) [5 marks] Consider a group G, two inequivalent irreps R_1 and R_2 of G with dimensions n_1 and n_2 and the reducible representation R defined by the block matrices

$$R(g) = \begin{pmatrix} R_1(g) & 0 & 0 & 0 & 0 \\ 0 & R_1(g) & 0 & 0 & 0 \\ 0 & 0 & R_1(g) & 0 & 0 \\ 0 & 0 & 0 & R_2(g) & 0 \\ 0 & 0 & 0 & 0 & R_2(g) \end{pmatrix} .$$

   Find the most general form of matrices M (with size $(3n_1 + 2n_2) \times (3n_1 + 2n_2)$) which satisfy $[M, R(g)] = 0$ for all $g \in G$.

   (c) [4 marks] For a group G and a sub-group $H \subset G$ the commutant $C_G(H)$ of H in G is defined by $C_G(H) = \{g \in G \,|\, gh = hg \ \ \forall h \in H\}$. Show that the commutant is a sub-group of G.

   (d) [5 marks] Consider the group $SU(5)$ and its $U(1)$ sub-group defined by the embedding $U(1) \ni z \mapsto \text{diag}(z^{-2}, z^{-2}, z^{-2}, z^3, z^3) \in SU(5)$. Find the commutant $C_{SU(5)}(U(1))$ of this $U(1)$ in $SU(5)$.

   (e) [7 marks] Write down the Young tableaux, tensors and highest-weight Dynkin labels for the $SU(5)$ representations $\mathbf{5}$, $\bar{\mathbf{5}}$ and $\mathbf{10}$. How do these representations branch under the sub-group $C_{SU(5)}(U(1)) \subset SU(5)$ found in part (d)?

4. (a) [5 marks] Write down the Dynkin diagram and the Cartan matrix for the algebra A_3. What are the weight systems of the A_3 irreps with highest weight Dynkin labels $(1, 0, 0)$ and $(0, 0, 1)$?

   (b) [6 marks] Write down the Dynkin diagram and the Cartan matrix for D_5 and find the weight system of the irrep with highest weight Dynkin label $(0, 0, 0, 0, 1)$.

   (c) [6 marks] Argue from the (extended) Dynkin diagram that $A_1 \oplus A_1 \oplus A_3$ is a maximal sub-algebra of D_5. The projection matrix $P = P(A_1 \oplus A_1 \oplus A_3 \subset D_5)$ is given by

$$P = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 \end{pmatrix} ,$$

   where the first two entries of a vector Pw (where w is a D_5 weight) correspond to the two A_1 Dynkin labels and the last three entries to the A_3 Dynkin label. Use this projection matrix to find the branching of the D_5 irrep from part (b) under the sub-algebra $A_1 \oplus A_1 \oplus A_3$. Denote the $A_1 \oplus A_1 \oplus A_3$ representation obtained in this way by R.

   (d) [8 marks] One family of the standard model of particle physics resides in the $SU_w(2) \times SU_c(3) \times U_Y(1)$ representation $R_F = (\mathbf{2}, \mathbf{3})_1 \oplus (\mathbf{1}, \bar{\mathbf{3}})_{-4} \oplus (\mathbf{1}, \bar{\mathbf{3}})_2 \oplus (\mathbf{2}, \mathbf{1})_{-3} \oplus (\mathbf{1}, \mathbf{1})_6$, where the subscript denotes the $U_Y(1)$ charge. Show that, for a suitable embedding of $SU_w(2) \times SU_c(3)$ into $SU(2) \times SU(2) \times SU(4)$, the representation R from part (c) branches into a representation which contains all $SU_w(2) \times SU_c(3)$ representations in R_F. Next, find an embedding of $U_Y(1)$ into $SU(2) \times SU(2) \times SU(4)$ such that the $U_Y(1)$ charges in R_F are reproduced correctly. What is a possible interpretation of the $SU_w(2) \times SU_c(3) \times U_Y(1)$ representation which is contained in R but not in R_F?