A15757W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM THEORY: PATH INTEGRALS AND MANY-PARTICLE PHYSICS

Trinity Term 2023

Thursday, 8th June 09:30 am - 11:30 am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. [25 marks] A one-dimensional model in classical statistical mechanics is defined as follows. Microscopic variables σ_n live on the sites n of a lattice, and can take any of Q possible states denoted by $\sigma_n = 1, 2 \ldots Q$. There is an interaction energy between neighbouring sites, which is $-J$ (with $J > 0$) if they are both in the same state and is zero otherwise. The total energy of a configuration for a lattice of L sites with periodic boundary conditions is hence

$$E = -J \sum_{n=1}^{L} \delta_{\sigma_n, \sigma_{n+1}} \tag{1}$$

where $\delta_{\sigma_n, \sigma_{n+1}}$ is the Kronecker delta and $\sigma_{L+1} \equiv \sigma_1$.

(a) [8 marks] What is the energy of (i) the ground states and (ii) the first excited states of this model? How many ground states are there, and how many first excited states?

(b) [10 marks] Explain how the transfer matrix method can be used to calculate the partition function Z for this model at inverse temperature β and show that the transfer matrix can be written in the form

$$\mathbb{T} = g(\beta J)\mathbb{I} + \mathbb{P}, \tag{2}$$

where $g(\beta J)$ is a scalar function which you should determine, $\mathbb{I}$ is the $Q \times Q$ unit matrix and $\mathbb{P}$ is the $Q \times Q$ matrix that has all entries $\mathbb{P}_{mn} = 1$. Evaluate $\mathbb{P}^2$ and $\operatorname{Tr} \mathbb{P}$. Hence or otherwise find the eigenvalues of $\mathbb{P}$ and of $\mathbb{T}$ and their degeneracies.

(c) [7 marks] Give an expression for the free energy $F = -\beta^{-1} \ln Z$ for finite L in the low-temperature regime, accurate to second order in the variable $e^{-\beta J}$. Explain how this expression can be understood in terms of low-energy configurations of the model.

2. [25 marks] Operators for the angular momentum components of a quantum spin are denoted by S^x, S^y and S^z. The total angular momentum operator is $S^2 = [S^x]^2 + [S^y]^2 + [S^z]^2$ and the spin raising and lowering operators are $S^\pm = S^x \pm iS^y$. They obey the commutation relations $[S^+, S^-] = 2S^z$ and $[S^z, S^\pm] = \pm S^\pm$. The states $|s,m\rangle$ are eigenstates of S^2 and S^z with eigenvalues $s(s+1)$ and m respectively (using units in which $\hbar = 1$).

The operators $a^\dagger$ and $b^\dagger$ are creation operators for two species of boson, with a and b the corresponding annihilation operators. They obey the commutation relations $[a, a^\dagger] = [b, b^\dagger] = 1$ and $[a, b] = [a, b^\dagger] = 0$. The states $|n_a, n_b\rangle$ are eigenstates of the boson number operators $a^\dagger a$ and $b^\dagger b$ with eigenvalues n_a and n_b respectively.

A quantum spin model (the XY ferromagnet) is defined as follows. Spins with $S_n^2 = s(s+1)$ are located at the sites n of a one-dimensional lattice and are represented by operators S_n^x, S_n^y and S_n^z. The Hamiltonian (with $J > 0$) is

$$H = -J \sum_{n=0}^{L-1} [S_n^x S_{n+1}^x + S_n^y S_{n+1}^y] . \tag{1}$$

The lattice consists of L sites and has periodic boundary conditions so that $S_L^\alpha \equiv S_0^\alpha$ for $\alpha = x, y$ and z.

(a) [9 marks] One way (out of several alternatives) to express spin operators in terms of boson operators is via the substitutions

$$S^+ = a^\dagger b, \quad S^- = b^\dagger a \quad \text{and} \quad S^z = \frac{1}{2}(a^\dagger a - b^\dagger b) . \tag{2}$$

Justify this by evaluating appropriate commutators. What is the correspondence between the set of states $|s,m\rangle$ and the set $|n_a, n_b\rangle$?

(b) [8 marks] Write the Hamiltonian H as a quartic in boson operators $a_n^\dagger$, $b_n^\dagger$, a_n and b_n defined at each site of the lattice. Fourier transformed operators are

$$A_q = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{iqn} a_n \quad \text{and} \quad B_q = \frac{1}{\sqrt{L}} \sum_{n=0}^{L-1} e^{iqn} b_n \tag{3}$$

with $q = 2\pi m/L$ and $m = 0, 1 \ldots L-1$. You may assume without proof that these operators satisfy the commutation relations $[A_q, A_k^\dagger] = [B_q, B_k^\dagger] = \delta_{qk}$ and $[A_q, B_k^\dagger] = [A_q, B_k] = 0$. Give with justification the inverse Fourier transforms. Express (i) $\frac{1}{2}(a_n^\dagger a_n + b_n^\dagger b_n)$ and (ii) the Hamiltonian H in terms of the operators A_q, $A_q^\dagger$, B_q and $B_q^\dagger$.

(c) [8 marks] Let $|\text{vac}\rangle$ denote the vacuum for the boson operators on the lattice. An approximate ground state wavefunction of H has the form

$$|\text{G}\rangle = \frac{1}{\sqrt{n_A! n_B!}} [A_0^\dagger]^{n_A} [B_0^\dagger]^{n_B} |\text{vac}\rangle . \tag{4}$$

You may assume without proof that $\langle G|G\rangle = 1$. Evaluate $\langle \text{G}|A_q^\dagger A_k|\text{G}\rangle$, giving the dependence on q and k. Find $\langle \text{G}|(a_n^\dagger a_n + b_n^\dagger b_n)|\text{G}\rangle$ and $\langle \text{G}|H|\text{G}\rangle$. Hence suggest the best choice for the integer parameters n_A and n_B to approximate the ground state. Evaluate the averages $\langle \text{G}|\hat{S}_m^x|\text{G}\rangle$ and $\langle \text{G}|\hat{S}_m^x \hat{S}_{m+n}^x|\text{G}\rangle$ for $n \neq 0$. Use these results to discuss briefly the physical interpretation of the state $|\text{G}\rangle$ in terms of the spin degrees of freedom, making reference to the concept of long-range order.